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# Generation of multipole moments by external field in Born–Infeld nonlinear electrodynamics

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**Abstract.** The mechanism for the generation of multipole moments due to an external field is presented for the Born–Infeld charged particle. The 'polarizability coefficient' $\kappa_l$ for arbitrary $l$-pole moment is calculated. It turns out that $\kappa_l \sim r_0^{2l+1}$, where $r_0 := \sqrt{|e|/4\pi b}$ and $b$ is the Born–Infeld nonlinearity constant. Some physical implications are considered.

## 1. Introduction

Recently, one of us proposed a consistent, relativistic theory of the classical Maxwell field interacting with classical, charged, point-like particles [1]. For this purpose an 'already renormalized' formula for the total four-momentum of a system composed of both the moving particles and the surrounding electromagnetic field was used. It was proved that the conservation of the total four-momentum defined by this formula is equivalent to a certain boundary condition on the behaviour of the Maxwell field in the vicinity of the particle trajectories. Without this condition, Maxwell theory with point-like sources is not dynamically closed: the initial conditions for particles and fields do not uniquely imply the future and the past of the system. Indeed, the particle trajectories fulfilling the initial conditions can be chosen *arbitrarily* and then the initial-value problem for the field alone can be solved uniquely. The boundary condition derived this way was called the *fundamental equation*. When added to the Maxwell equations, it provides the missing dynamical equation: now, particles trajectories cannot be chosen arbitrarily and initial data uniquely imply the future and the past evolution of the 'particles + fields' system.

Physically, the 'already renormalized' formula for the total four-momentum was suggested by a suitable approximation procedure applied to an extended-particle model. In such a model we suppose that the particle is a stable, soliton-like solution of a hypothetical fundamental theory of interacting electromagnetic and matter fields. We assume that this hypothetical theory tends asymptotically to the linear Maxwell electrodynamics, in the weak field regime (i.e. for weak electromagnetic fields and 'almost vanishing' matter fields). This means that 'outside of the particles' the entire theory reduces to Maxwell electrodynamics. Starting from this model, a formula was found, which gives in a good approximation the total four-momentum of the system composed of both the moving (extended) particles and the surrounding electromagnetic field. This formula uses only

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the 'mechanical' information about the particle (position, velocity, mass $m$ and electric charge $e$) and the free electromagnetic field outside of the particle. It turns out that this formula does not produce any infinities when applied to the case of point particles, i.e. it is 'already renormalized'. Using this philosophy, this formula was taken as a starting point for a mathematically self-consistent theory of *point-like particles* interacting with the linear Maxwell field. The 'fundamental equation' of the theory is precisely the conservation of the total four momentum of the particles + fields system defined by this formula.

At this point a natural idea arises, to construct a 'second-generation' theory, which better approximates the real properties of an extended particle, and also takes into account possible deformations of its interior, due to the strong external field. In [2] a simple mechanism for the generation of the electric dipole moment of a particle was proposed. As a specific model for the particle we have used the Born–Infeld particle described by the Born–Infeld nonlinear electrodynamics.

In the present paper we prove that a similar mechanism is responsible for the generation of higher multipole moments.

Mathematically, such a polarizability is due to the elliptic properties of the field equations describing the statics of the physical system under consideration. Given a particular model of the matter fields interacting with electromagnetism, the 'particle-at-rest' solution corresponds to a minimum of the total field energy. It is, therefore, described by a solution of a system of elliptic equations (Euler–Lagrange equations derived from the total Hamiltonian of the hypothetical fundamental theory of interacting matter fields and electromagnetic field). Far away from the particle, these equations reduce to the free Maxwell equations.

This solution corresponds to the vanishing boundary conditions at infinity. The physical situation of a 'particle in a non-vanishing external field' corresponds to the solution of *the same* elliptic equations but with *non-vanishing* boundary conditions $\boldsymbol{E}_\infty$ at infinity. More precisely, we assume that the electric field at infinity varies as

$$E^k(\boldsymbol{x}) = E^k_{\mathrm{reg}}(\boldsymbol{x}) + E^k_\infty(\boldsymbol{x}) \tag{1}$$

where

$$E^k_\infty(\boldsymbol{x}) := \mathcal{Q}^k{}_{i_2\ldots i_l} x^{i_2}\ldots x^{i_l} \tag{2}$$

(the $l$-pole tensor $\mathcal{Q}_{i_1 i_2\ldots i_l}$ is completely symmetric and traceless) and the regular part $E^k_{\mathrm{reg}}(\boldsymbol{x})$ vanishes at infinity.

Suppose that the free particle (the unperturbed solution) possesses no internal structure. This means that for vanishing external field $\mathcal{Q} = 0$ the regular part $E^k_{\mathrm{reg}}$ reduces to the simplest Coulomb field describing the monopole with a given electric charge $e$. However, for a non-trivial perturbation $\mathcal{Q}$ the regular field may contain an extra multipole term at infinity, of the type

$$E^k_{\mathcal{M}} := \frac{r^2\mathcal{M}^k{}_{i_2\ldots i_l}x^{i_2}\ldots x^{i_l} - \big((2l+1)/l\big)x^k\mathcal{M}_{i_1 i_2\ldots i_l}x^{i_1}x^{i_2}\ldots x^{i_l}}{r^{3+2l}}. \tag{3}$$

The reason for the creation of this extra multipole moments is the nonlinearity of the field in the strong-field region. For weak perturbations, the relation between the $l$-pole moment $\mathcal{M}_{i_1 i_2\ldots i_l}$ of the particle created this way and the $l$-pole moment $\mathcal{Q}_{i_1 i_2\ldots i_l}$ of the external field is expected to be linear in the first approximation:

$$\mathcal{M}_{i_1 i_2\ldots i_l} = \kappa_l\, \mathcal{Q}_{i_1 i_2\ldots i_l} \tag{4}$$

and the coefficient $\kappa_l$ describes the 'deformability' of the particle, due to nonlinear character of the interaction of the matter fields (constituents of the particle) with the electromagnetic field.

The coefficients $\kappa_l$ arise, therefore, in a manner similar to 'reflection' or 'transmission' coefficients in scattering theory. Equations (2) and (3) describe two independent solutions of the second order, linear, elliptic equation describing the free, statical Maxwell field surrounding the particle. Outside the particle they may be mixed in arbitrary proportions. Such an arbitrary mixture is no longer possible if it has to match an exact solution of nonlinear equations describing the interior of the particle. Relation (4) arises, therefore, as the matching condition between these two solutions.

In the present paper we assume that the unperturbed particle is described by the spherically symmetric, static solution of nonlinear Born–Infeld electrodynamics with a $\delta$-like source. We find the two-dimensional family of all the $l$-pole perturbations of the above solution explicitly. They all behave correctly at $r \to \infty$. For $r \to 0$, however, there is one perturbation which remains regular, and another one which increases faster then the unperturbed solution. The variation of the total field energy due to the latter perturbation is divergent in the vicinity of the particle, which we consider to be an unphysical feature. We conclude that all the physically admissible perturbations must be proportional to the one which is regular at 0. At $r \to \infty$ this solution behaves like a mixture of the solutions (2) and (3). We calculate the ratio between these two ingredients and we interpret it as the $l$th polarizability coefficient of the Born–Infeld particle.

## 2. Perturbations of the Born–Infeld particle

Born–Infeld electrodynamics [3] (see also [4]) are defined by the following Lagrangian:

$$\mathcal{L}_{\mathrm{BI}} := b^2 \left[1 - \sqrt{1 - 2b^{-2}S - b^{-4}P^2}\right] \tag{5}$$

where $S$ and $P$ are the following Lorentz invariants:

$$S := -\tfrac{1}{4} f_{\mu\nu} f^{\mu\nu} = \tfrac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) \tag{6}$$

$$P := -\tfrac{1}{4} \epsilon_{\mu\nu\lambda\kappa} f^{\mu\nu} f^{\lambda\kappa} = \boldsymbol{E}\boldsymbol{B} \tag{7}$$

and $f_{\mu\nu}$ is a tensor of the electromagnetic field defined in a standard way by a four-potential vector. The parameter '$b$' has the dimensions of field strength (Born and Infeld called it the *absolute field* [3]) and it measures the nonlinearity of the theory. In the limit $b \to \infty$ the Lagrangian $\mathcal{L}_{\mathrm{BI}}$ tends to the standard Maxwell Lagrangian

$$\mathcal{L}_{\mathrm{Maxwell}} = S. \tag{8}$$

Note that field equations derived from (5) have the same form as the Maxwell equations derived from (8), i.e.

$$\nabla \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{9}$$

$$\nabla \times \boldsymbol{H} - \dot{\boldsymbol{D}} = \boldsymbol{j} \qquad \nabla \cdot \boldsymbol{D} = \rho \tag{10}$$

(to obtain the Born–Infeld equations with sources $\rho$ and $\boldsymbol{j}$ one has to add to $\mathcal{L}_{\mathrm{BI}}$ the standard interaction Lagrangian '$j^\mu A_\mu$'). However, the relation between fields $(\boldsymbol{E}, \boldsymbol{B})$ and $(\boldsymbol{D}, \boldsymbol{H})$ is now highly nonlinear:

$$\boldsymbol{D} := \frac{\partial \mathcal{L}_{\mathrm{BI}}}{\partial \boldsymbol{E}} = \frac{\boldsymbol{E} + b^{-2}(\boldsymbol{E}\boldsymbol{B})\boldsymbol{B}}{\sqrt{1 - b^{-2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) - b^{-4}(\boldsymbol{E}\boldsymbol{B})^2}} \tag{11}$$

$$\boldsymbol{H} := -\frac{\partial \mathcal{L}_{\mathrm{BI}}}{\partial \boldsymbol{B}} = \frac{\boldsymbol{B} - b^{-2}(\boldsymbol{E}\boldsymbol{B})\boldsymbol{E}}{\sqrt{1 - b^{-2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) - b^{-4}(\boldsymbol{E}\boldsymbol{B})^2}}. \tag{12}$$

The above formulae are responsible for the nonlinear character of the Born–Infeld theory. In the limit $b \to \infty$ we obtain linear Maxwell relations: $\boldsymbol{D} = \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}$ (we use the Heaviside–Lorentz system of units).

Now, consider a point-like, Born–Infeld charged particle at rest. It is described by the static solution of the Born–Infeld field equations (9)–(12) with $\rho = e\delta(\boldsymbol{r})$ and $\boldsymbol{j} = 0$, where $e$ denotes the electric charge of the particle. Obviously $\boldsymbol{B} = \boldsymbol{H} = 0$ (Born–Infeld electrostatics). Moreover, the spherically symmetric solution of $\nabla \cdot \boldsymbol{D} = e\delta(\boldsymbol{r})$ is given by the Coulomb formula

$$\boldsymbol{D}_0 = \frac{e}{4\pi r^3}\boldsymbol{r}. \tag{13}$$

Using equation (11) one can easily find the corresponding $\boldsymbol{E}_0$ field:

$$\boldsymbol{E}_0 = \frac{\boldsymbol{D}_0}{\sqrt{1 + b^{-2}\boldsymbol{D}_0^2}} = \frac{e}{4\pi r}\frac{\boldsymbol{r}}{\sqrt{r^4 + r_0^4}} \tag{14}$$

where we have introduced $r_0 := \sqrt{|e|/4\pi b}$. Note that the field $\boldsymbol{E}_0$, in contrast to $\boldsymbol{D}_0$, is bounded in the vicinity of a particle: $|\boldsymbol{E}_0| \leqslant |e|/4\pi r_0^2 = b$. It implies that the energy of a point charge is already finite.

Now, let us perturb the static Born–Infeld solution $\boldsymbol{E}_0$:

$$\boldsymbol{E} = \boldsymbol{E}_0 + \widetilde{\boldsymbol{E}} \tag{15}$$

where $\widetilde{\boldsymbol{E}}$ denotes a weak perturbation, i.e. $|\widetilde{\boldsymbol{E}}| \ll |\boldsymbol{E}_0|$. The corresponding $\boldsymbol{D}$ field may be obtained from (11). In the electrostatic case, i.e. $\boldsymbol{B} = \boldsymbol{H} = 0$, equation (11) reads:

$$\boldsymbol{D} = \frac{\boldsymbol{E}}{\sqrt{1 - b^{-2}\boldsymbol{E}^2}} \qquad \boldsymbol{E} = \frac{\boldsymbol{D}}{\sqrt{1 + b^{-2}\boldsymbol{D}^2}}. \tag{16}$$

Therefore, using equation (15) we obtain

$$\boldsymbol{D} = \boldsymbol{D}_0 + \frac{1}{\sqrt{1 - b^{-2}\boldsymbol{E}_0^2}}\left(\widetilde{\boldsymbol{E}} + b^{-2}(\boldsymbol{D}_0\widetilde{\boldsymbol{E}})\boldsymbol{D}_0\right) + \mathrm{O}(\widetilde{\boldsymbol{E}}^2) \tag{17}$$

where $\mathrm{O}(\widetilde{\boldsymbol{E}}^2)$ denotes terms vanishing for $|\widetilde{\boldsymbol{E}}| \to 0$ as $\widetilde{\boldsymbol{E}}^2$ or faster. In the present paper we study only the *linear perturbation*, i.e. we keep in (17) terms linear in $\widetilde{\boldsymbol{E}}$ and neglect $\mathrm{O}(\widetilde{\boldsymbol{E}}^2)$. Therefore, in this approximation the perturbation $\widetilde{\boldsymbol{D}} = \boldsymbol{D} - \boldsymbol{D}_0$ of the field $\boldsymbol{D}_0$ equals

$$\widetilde{\boldsymbol{D}} = \frac{1}{\sqrt{1 - b^{-2}\boldsymbol{E}_0^2}}\left(b^{-2}(\boldsymbol{D}_0\widetilde{\boldsymbol{E}})\boldsymbol{D}_0 + \widetilde{\boldsymbol{E}}\right) = \sqrt{1 + b^{-2}\boldsymbol{D}_0^2}\left(b^{-2}(\boldsymbol{D}_0\widetilde{\boldsymbol{E}})\boldsymbol{D}_0 + \widetilde{\boldsymbol{E}}\right). \tag{18}$$

The fields $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{D}}$ fulfill the following equations:

$$\nabla \times \widetilde{\boldsymbol{E}} = 0 \qquad \nabla \cdot \widetilde{\boldsymbol{D}} = 0. \tag{19}$$

The first implies that $\widetilde{\boldsymbol{E}} = -\nabla\widetilde{\phi}$. Therefore, using equation (18), $\nabla \cdot \widetilde{\boldsymbol{D}} = 0$ leads to the following equation for the potential $\widetilde{\phi}$:

$$\Delta\widetilde{\phi} + \left(\frac{r_0}{r}\right)^4 \left[\frac{\partial^2\widetilde{\phi}}{\partial r^2} - \frac{4}{r}\frac{\partial\widetilde{\phi}}{\partial r}\right] = 0 \tag{20}$$

where $\Delta$ stands for a three-dimensional Laplace operator in $\mathbb{R}^3$. Observe that in the limit $r_0 \to 0$ (i.e. Maxwell theory) we obtain simply the Laplace equation $\Delta\widetilde{\phi} = 0$ for the electrostatic potential $\widetilde{\phi}$. Using spherical coordinates in $\mathbb{R}^3$ the Laplace operator $\Delta$ reads:

$$\Delta\widetilde{\phi} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\widetilde{\phi}) + \frac{1}{r^2}\boldsymbol{L}^2\,\widetilde{\phi} \tag{21}$$

where $\boldsymbol{L}^2$ denotes the Laplace–Beltrami operator on the unit sphere (it is equal to the square of the quantum-mechanical angular momentum).

We see that due to the spherical symmetry of the unperturbed solution $\boldsymbol{D}_0$, different multipole modes decouple in the above equation. Therefore, any solution of (20) may be written as follows:

$$\widetilde{\phi}(\boldsymbol{x}) = \sum_{l=1}^{\infty} a_l \widetilde{\phi}_l(\boldsymbol{x}) \tag{22}$$

where

$$\widetilde{\phi}_l(r, \text{angles}) := \frac{\Psi_l(r)}{r}\; Y_l(\text{angles}) \tag{23}$$

and $Y_l$ denotes the $l$-pole eigenfunction of $\boldsymbol{L}^2$, i.e. $\boldsymbol{L}^2 Y_l = -l(l+1)Y_l$. Obviously, an eigenfuction $Y_l$ is related to the $l$-pole moment tensor by

$$Y_l = r^{-l}\, x^{i_1} \ldots x^{i_l}\, \mathcal{Q}_{i_1 i_2 \ldots i_l}. \tag{24}$$

Observe that we do not consider the monopole term (i.e. $a_0\, \widetilde{\phi}_0$) in (22). This term corresponds to the gauge transformation of $\widetilde{\phi}$ and due to the gauge invariance of (20) it is inessential (we may fix the gauge by putting, e.g., $a_0 = 0$).

Let us look for the $l$-pole-like deformation, i.e. for a function $\widetilde{\phi}_l$. Inserting the ansatz (23) in equation (20) we obtain the following equation for $\Psi_l(r)$:

$$\left(\Psi_l'' - \frac{l(l+1)}{r^2}\Psi_l\right) + \left(\frac{r_0}{r}\right)^4 \left(\Psi_l'' - \frac{6}{r}\Psi_l' + \frac{6}{r^2}\Psi_l\right) = 0 \tag{25}$$

where $\Psi_l'$ stands for $\partial\Psi_l/\partial r$. In section 3 we find the two-dimensional space of solutions of (25).

## 3. Exact solution of the ‘deformed’ Laplace equation

Let us note that for $r \gg r_0$, equation (20) reduces to the standard Laplace equation with two independent $l$-pole-like solutions: the one corresponding to the constant $l$-pole moment (it varies as $r^{l+1}$) and the external $l$-pole solution (varying as $r^{-l}$).

On the other hand, a basis may be chosen corresponding to the behaviour of $\Psi_l$ at $r \to 0$. From the asymptotic analysis of (25) it follows that there is a solution which varies as $r^6$ and another one which varies as $r$. Let us define $\Phi_l := r^{-6}\Psi_l$ and introduce the following variable:

$$z := -\left(\frac{r}{r_0}\right)^4. \tag{26}$$

Using (25) one obtains the following equation for $\Phi_l$:

$$z(1-z)\frac{\mathrm{d}^2\Phi_l}{\mathrm{d}z^2} + \left(\frac{9}{4} - \frac{15}{4}z\right)\frac{d\Phi_l}{dz} - \frac{30 - l(l+1)}{16}\Phi_l = 0. \tag{27}$$

This is a hypergeometric equation [5]. A hypergeometric equation for a function $u = u(z)$

$$z(1-z)\,u'' + [c - (a+b+1)]\,u' - ab\,u = 0 \tag{28}$$

has two independent solutions. One of them is given by the hypergeometric function ${}_2F_1(a, b, c, z)$. The other one has the following form (for $c \neq 1$):

$$z^{1-c}\,{}_2F_1(b-c+1, a-c+1, 2-c, z). \tag{29}$$

Therefore, the general solution of (25) reads

$$\Psi_l(r) = A_l\, r^6\, {}_2F_1\left(\frac{l+6}{4}, \frac{5-l}{4}, \frac{9}{4}, -\left(\frac{r}{r_0}\right)^4\right) + B_l\, r\, {}_2F_1\left(\frac{l+1}{4}, -\frac{l}{4}, -\frac{1}{4}, -\left(\frac{r}{r_0}\right)^4\right). \tag{30}$$

The first term on the right-hand side of (30) at $r \to 0$ varies as $r^6$, the other one as $r$.

Let us note, however, that the second term (varying as $r$) corresponds to the unphysical solution. To see this let us look for the behaviour of the electric field $\widetilde{\boldsymbol{E}}$ 'produced' by it in the vicinity of the Born–Infeld particle, i.e. for $r \to 0$. From equations (23) and (24) we obtain

$$\widetilde{E}_k = -\partial_k \widetilde{\phi} = -\frac{l}{r}[x \cdot \mathcal{Q}]_m \left(\delta^m_{\ k} - \frac{x^m x_k}{r^2}\right) + l(l+1)\frac{r^3}{4r_0^4}\left(\frac{4-l}{r}[x \cdot \mathcal{Q}]x_k + l[x \cdot \mathcal{Q}]_k\right) + \mathrm{O}(r^7) \tag{31}$$

where

$$[x \cdot \mathcal{Q}] := r^{-l} x^{i_1} \ldots x^{i_l} \mathcal{Q}_{i_1 \ldots i_l} \tag{32}$$

$$[x \cdot \mathcal{Q}]_k := r^{-l+1} x^{i_1} \ldots x^{i_{l-1}} \mathcal{Q}_{i_1 \ldots i_{l-1} k}. \tag{33}$$

Therefore, due to the first term in (31), $\widetilde{\boldsymbol{E}}$ exceeds the unperturbed field $\boldsymbol{E}_0$ itself.

Moreover, the 'perturbation' $\widetilde{\boldsymbol{E}}$ leads to infinite variation of the total field energy. The 'electrostatic' energy $\mathcal{H}$ corresponding to electric induction $\boldsymbol{D}$ is given [3, 4] by

$$\mathcal{H} = \int b^2 \left(\sqrt{1 + b^{-2}\boldsymbol{D}^2} - 1\right) \mathrm{d}^3x. \tag{34}$$

Therefore, its variation reads

$$\delta\mathcal{H}|_{\boldsymbol{D}_0} = \int \frac{D_0^k \delta D_k}{\sqrt{1 + b^{-2}\boldsymbol{D}_0^2}}\, \mathrm{d}^3x = \int E_0^k \delta D_k\, \mathrm{d}^3x = \int E_0^k \widetilde{D}_k\, \mathrm{d}^3x. \tag{35}$$

Now, let us investigate the behaviour of $\boldsymbol{E}_0 \widetilde{\boldsymbol{D}}$ for $r \to 0$. From (18) it follows that this expression contains highly singular term which varies as $r^{-6} \times$ radial component of $\widetilde{\boldsymbol{E}}$. Using the expansion (31) we see that the first term is purely tangential, i.e. it is orthogonal to any radial vector. However, the second one does contain a radial part $(e/4\pi)r^3 l(l+1)[x \cdot \mathcal{Q}]$ and this way $\boldsymbol{E}_0 \widetilde{\boldsymbol{D}}$ produces non-integrable singularity $\sim r^{-3}[x \cdot \mathcal{Q}]$. Therefore, we conclude that the solution varying as $r$ for $r \to 0$ has to be excluded from our consideration, i.e. $B_l = 0$. This way the solution of (25) is given by

$$\Psi_l(r) = A_l\, r^6\, {}_2F_1\left(\frac{l+6}{4}, \frac{5-l}{4}, \frac{9}{4}, -\left(\frac{r}{r_0}\right)^4\right). \tag{36}$$

## 4. Multipole moments of the particle

Knowing the exact solution of (25), we are ready to calculate the $l$th deformability coefficient of the Born–Infeld particle. We know that at infinity, i.e. for $r \gg r_0$, $\Psi_l$ is a combination of two solutions: one varying as $r^{l+1}$ (it corresponds to the constant $l$-pole field) and the other varying as $r^{-l}$ (corresponding to the external $l$-pole solution). Due to the fact that the space of physically admissible solutions is only one dimensional (see equation (36)), the proportion between these two solutions is no longer arbitrary. The ratio between them therefore has physical meaning. To find this ratio we shall use the following property of a hypergeometric function ${}_2F_1(a, b, c, z)$ [5]:

$$
\begin{aligned}
{}_2F_1(a, b, c, z) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\left(a, a+1-c, a+1-b, \frac{1}{z}\right) \\
& + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left(b, b+1-c, b+1-a, \frac{1}{z}\right)
\end{aligned} \tag{37}
$$

where $\Gamma$ denotes the Euler $\Gamma$-function. This identity allows us to find the asymptotic behaviour of $\Psi_l$ at infinity. Using equations (36) and (37) one immediately obtains

$$
\begin{aligned}
\Psi_l(r) = {} & X_l\, r^{l+1}\,{}_2F_1\left(-\frac{l}{4}, \frac{5-l}{4}, \frac{3-2l}{4}, -\left(\frac{r_0}{r}\right)^4\right) + \\
& + Y_l\, r^{-l}\,{}_2F_1\left(\frac{l+6}{4}, \frac{l+1}{4}, \frac{5+2l}{4}, -\left(\frac{r_0}{r}\right)^4\right)
\end{aligned} \tag{38}
$$

where

$$
X_l = A_l\, \frac{\Gamma(\frac{9}{4})\Gamma(-\frac{1}{4}(2l+1))}{\Gamma(\frac{1}{4}(l+6))\Gamma(\frac{1}{4}(l+4))}\, r_0^{5-l} \tag{39}
$$

$$
Y_l = A_l\, \frac{\Gamma(\frac{9}{4})\Gamma(\frac{1}{4}(2l+1))}{\Gamma(\frac{1}{4}(5-l))\Gamma(\frac{1}{4}(3-l))}\, r_0^{6+l}. \tag{40}
$$

Note that the first term on the right-hand side of (38) at infinity varies as $r^{l+1}$ and corresponds to the constant $l$-pole field $\boldsymbol{E}_\infty$ given by (2). The second term varies as $r^{-l}$ and corresponds to the external $l$-pole solution $\boldsymbol{E}_\mathcal{M}$ given by (3). We interpret the ratio between these two ingredients in equation (38):

$$
\kappa_l := \frac{Y_l}{X_l} \tag{41}
$$

as the 'deformability coefficient' (more precisely, the $l$th deformability coefficient) of the Born–Infeld particle. It measures the $l$-pole moment $\mathcal{M}_{i_1\ldots i_l}$ of the particle generated by the constant $l$-pole moment $\mathcal{Q}_{i_1\ldots i_l}$ of the external electric field. Using equations (39) and (40) we obtain

$$
\kappa_l = \frac{\Gamma(\frac{1}{4}(l+6))\Gamma(\frac{1}{4}(l+4))\Gamma(-\frac{1}{4}(2l+1))}{\Gamma(\frac{1}{4}(5-l))\Gamma(\frac{1}{4}(3-l))\Gamma(\frac{1}{4}(2l+1))}\, r_0^{2l+1}. \tag{42}
$$

Obviously, in the limit of Maxwell theory ($r_0 \to 0$), the external electric field does not generate the $l$-pole moment of any particle. Therefore, this mechanism comes entirely from the nonlinearity of the Born–Infeld theory.

## 5. Physical implications

In this paper we have used a very specific model of nonlinear electrodynamics. There is a natural question: why this particular model? It turns out that among other nonlinear theories of electromagnetism, the Born–Infeld theory possesses very distinguished physical properties [6]. For example, it is the only causal spin-1 theory [7] (apart from the Maxwell theory). The assumption that the theory is effectively nonlinear in the vicinity of a charged particle is very natural from the physical point of view. Actually, we have learned this from quantum electrodynamics. There have been attempts to identify the non-polynomial Born–Infeld Lagrangian as an effective Euler–Heisenberg Lagrangian [8]. It has been shown [9] that the effective Lagrangian can coincide with equation (5) up to six-photon interaction terms. Recently, there has been renewed interest in Born–Infeld electrodynamics due to the investigation in the string theory (see, e.g., [10]), where equation (5) was not postulated, but was derived.

Now, let us make some comments concerning physical implications of the obtained results. First of all taking $l = 1$ one obtains

$$\kappa_1 = -1.854\,07\, r_0^3 \tag{43}$$

which reproduces the result obtained in [2]. In [2] it was mentioned that for $l = 1$ one might describe the polarizability coefficient of the proton in this way. According to [11], we have $\kappa_1 = (12.1 \pm 0.9) \times 10^{-4}$ fm$^3$. To fit this value one has to take $r_0 \approx 0.09$ fm. The total mass of the corresponding Born–Infeld unperturbed field accompanying such a particle is about 32 electron masses. We see that the main part of the proton total mass cannot be of electromagnetic nature and has to be concentrated in the material core of the particle.

Unfortunately, there are no experimental data concerning particle polarizability for $l \neq 1$. It will probably be a highly non-trivial task to measure these quantities experimentally. It would be very interesting to have the possibility of comparing such quantities with equation (42).

Let us note that $\kappa_3$ and $\kappa_5$ vanish due to $\Gamma(0)$ in the denominator of equation (42). All other $\kappa_l \neq 0$. It does not mean that the particle is not polarizable for $l = 3$ and $l = 5$. It is true in the linear approximation only. However, this result suggests that in these two sectors it is much more difficult to polarize the particle than in the other ones.

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